

APPLICATION OF BOUNDARY ELEMENT METHOD TO SOLUTION OF TRANSIENT HEAT CONDUCTION

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Abstract: The object of this paper is the implementation of boundary element method to solving the transient heat transfer problem with nonzero boundary condition and particularly with periodic boundary condition. The new mathematical BEM algorithm for two dimensional transient heat conduction problem with periodic boundary condition is developed and verified. The results of numerical simulation of transient heat conduction in two dimensional flat plate under non zero initial condition are compared with results obtained with analytical method. Then the practical application of developed algorithm is presented, that is the solution of ground temperature distribution problem with oscillating temperature of ambient. All results were obtained with a new authoring computer program for solving transient heat conduction problem, written in Fortran.

Keywords: Boundary Element Method, Transient Heat Conduction Problem, Periodic Boundary Condition

1. INTRODUCTION

The heat transfer in solids, with the changes of temperature in time on physical boundaries of analysed objects, occur in many engineering mechanisms (engines, compressors), heating and cooling systems and hydraulic networks (Zhang et al., 2009; Lu and Viljanen, 2006). The analysis of basic mechanism of heat transfer in solids, that is heat conduction problem, is significant for process of designing and optimization mechanical systems and devices. Accordingly, the heat conduction equations with conditions of variable temperature or heat flux on boundaries become an important instrument for mathematical description of many engineering, geothermal and biological problems. As a result, there is a need to develop effective computational methods and tools for solving transient heat conduction problem (Mansur et al., 2009; Yang and Gao, 2010).

Two groups of method are applied to obtain transient heat conduction problem solution: analytical and numerical. In the literature, many analytical methods have been proposed, inter alia based on orthogonal and quasi-orthogonal expansion technique, Laplace transform method, Green's function approach or finite integral transform technique, but they are feasible only for problems with simple geometries (Singh et al., 2008).

Monte et al. (2012) presented very accurate analytical solutions modeling transient heat conduction processes in 2D Cartesian finite bodies, such as rectangle and two layer objects, for small values of the time. In the paper, the geometry criterion was provided that permit to use 1D semi-infinite solutions for solving 2D finite single- and multi-layer transient heat conduction problems. Yumrutas (Yumrutas et al., 2005) developed new method based on complex finite Fourier transform (CFFT) technique for calculation of heat flux, through multilayer walls and flat roofs, and the temperature on the inner surface. The periodic boundary conditions were assumed, that is hourly changeable values of external air temperature and solar radiation. Lu et al. (Lu et al., 2006; Lu and Viljanen, 2006) adopted the Laplace transform to solve the multidimensional heat conduction in composite circular cylinder and multilayer sphere, with time-dependent temperature changes on boundary, which were approximated as Fourier

series. Singh et al. (2008) applied separation of variables method to obtain analytical solution, in the form of transient temperature distribution, to the 2D transient heat conduction problem in polar coordinates with multiple layers in the radial direction. Rantala (2005) proposed a new semi-analytical method for the calculation of temperature distribution along the fill layer underneath a slab-on-ground structure subjected to periodic external and internal temperature.

In spite of development of analytical techniques, this methods still cannot be employed for solving most practical heat transfer problems, such as heat conduction in anisotropic materials, objects of complex geometries or complex boundary conditions (Rantala, 2005; Johansson and Lesnic, 2009). Hence, for last few decades, the numerical methods have been strongly developed, as more universal computational tool.

The most popular are mesh methods, such as the finite element method (FEM) and the finite difference method (FDM). Although this methods are well established and commonly applied to transfer heat analysis, in many problems, mesh generation can be very laborious and constitutes the most expensive and difficult part of numerical simulations. Moreover, in objects of complex geometries, generated meshes can be distorted, what contributes to increase of computational error (Li, 2011).

The drawback of mesh generation is overcome in the mesh free (meshless) methods, that use a set of scattered nodal points in considered object (no connectivity among nodes), instead of meshes (Cheng and Liew, 2012; Ochiai et al., 2006). Some of this methods have been recently applied to transient heat conduction analysis in 2D objects, like meshless element free Galerkin (EFGM) method (Zhang et al., 2009), meshless local Petrov-Galerkin (MLPG) method (Li et al., 2011), method of fundamental solutions MFS (Johansson and Lesnic, 2008, 2009), meshless local radial basis function-based differential quadrature (RBF-DQ) method (Soleimani et al., 2010), and in 3D objects, like meshless reproducing kernel particle (RKPM) method (Cheng and Liew, 2012). The disadvantage of this methods, is that, in some cases, like transient heat conduction, they are more time-consuming than mesh methods, such as FEM, because of the larger dimensions of generated matrices (Zhang et al., 2009).

The alternative for above mentioned mesh and mesh free

methods is boundary element method (BEM). Compared to grid methods (FDM, FEM), the great advantage of BEM is the possibility of determination of the solution (both the function and the derivative of this function) at any point of the domain without necessity of construction of grids in considered 2D or 3D space. The discretization is performed only over the boundary, not over the whole analyzed domain hence the size of system of equations, that need to be solved, is reduced by one. In BEM, the fully populated coefficient matrices are generated, what is the opposite of banded and symmetric matrices in FEM. However, the small dimensions of BEM matrices counterbalance this disadvantage (Katsikadelis, 2002; Majchrzak, 2001; Pozrikidis, 2000). Application of the BEM requires the knowledge of fundamental solution of the governing differential operator, but at the same time, the use of fundamental solution stabilize the numerical commutations (Ochiai et al., 2006).

The BEM is successfully applied to steady and unsteady heat conduction problems. As opposed to steady problem, in mathematical description of transient heat conduction, the domain integrals occur. In order to keep the boundary character of the method, many different techniques have been developed, but the most popular are: method using the Laplace transformation to eliminate the time derivative, the dual reciprocity method, and the convolution scheme (employing time-dependent fundamental solutions).

Erhart (Erhart et al., 2006) implemented the Laplace transformation for solution of transient heat transfer in multi-region objects. As a result the time-independent boundary integral equation was produced, solved further with a steady BEM approach. The last step was numerical inversion of the solution, done with the use of Stehfest method. The derived algorithm was applied to heat conduction in a bar, laminar airfoil with three cooling passages and non-symmetric airfoil. The results were compared with those obtained with finite volume method (FVM).

Sutradhar and Paulino (2004) also used the Laplace transformation, both with Galerkin approximation, for analysis of the non-homogenous transient heat conduction problem in functionally graded materials FGM of variable thermal conductivity and specific heat. The three kinds of material variation, that is quadratic, exponential and trigonometric, were assumed for verifying the accuracy of presented method. The practical example for the functionally graded rotor problem was carried out.

Another approach is Fourier transform, applied by Simoes (Simoes et al., 2012) and Godinho (Godinho et al., 2004), consist in three general steps: converting analyzed domain into frequency domain, solving the heat conduction problem with BEM and obtaining the final solution in time domain with the use of inverse Fourier transform. Simoes tested method in 2D object with unit initial temperatures and with non-constant temperature distribution in domain. Godinho analyzed transient heat conduction around a cylindrical irregular inclusion of infinite length, inserted in a homogeneous elastic medium and subjected to heat point sources placed at some point in the host medium.

Mohammadia (Mohammadia et al., 2010) solved 2D nonlinear transient heat conduction problems with non-uniform and nonlinear heat sources, with the new BEM approach, using time-dependent fundamental solutions. In this method temperature is computed on the boundary and in internal points at every time step, and the results constitute the initial values for the next time step. However, for 3D and large problems, the storage of coefficient matrices for every time step can be problematic (Erhart et al., 2006).

Tanaka et al. (2006) applied dual reciprocity boundary element method (DRBEM) for analysis of 3D transient heat conduction problem in nonlinear temperature-dependent materials. In proposed method, domain integral is transformed into boundary integrals with the use of radial basis functions. To entertain the material nonlinearity, the iterative solution procedure was employed. Bialecki et al. (2002) proposed the DRBEM without matrix inversion for linear and non-linear transient heat conduction problem, that reduce the time of computations. The method was applied to solve heat transfer problem in a turbine rotor blade. Ochiai (Ochiai et al., 2006; Ochiai and Kitayama, 2009) developed the triple-reciprocity BEM to solve 2D and 3D transient heat conduction problems. One of the recent methods is radial integration boundary element method RIBEM applied to transient heat conduction problem by Yang and Gao (2010), which can be employed to analysis of functionally graded material problems.

In this paper, BEM is applied to solve the unsteady heat conduction problem in 2D area of arbitrary shape of boundary line in particular case of periodic changes of temperature on boundary line. The new mathematical BEM algorithm for periodic boundary condition was developed, both with a new authoring computer program, written in Fortran, applied to verifying the accuracy of presented algorithm and to solving practical example.

2. TRANSIENT HEAT CONDUCTION

The thermal processes, in which the heat conduction is the main mechanism, are described by Fourier-Kirchhoff equation.

The unsteady heat conduction in homogeneous solid substance with constant material properties without inner heat sources, is described by the heat conduction equation (also named thermal diffusion equation)

$$\left(\nabla^2 - \frac{1}{\alpha} \frac{\partial}{\partial t} \right) T(x, y, t) = 0 \quad (1)$$

In the above equation $\alpha = \lambda / c$ is the thermal diffusivity, in which λ is the thermal conductivity and c is the volumetric specific heat; and $\partial / \partial t$ is the local time derivative.

In order to find the solution of this equation, it is necessary to introduce the boundary conditions (1a) and (1b), and initial condition (1c) that take the following form:

$$T(x, y, t) = T_L(x, y, t) \quad , \quad (x, y) \in L_q \quad (1a)$$

$$q(x, y, t) = -\lambda \frac{\partial T(x, y, t)}{\partial n} = q_L(x, y, t) \quad , \quad (x, y) \in L_T \quad (1b)$$

$$T(x, y, 0) = T_0(x, y) \quad , \quad (x, y) \in \Lambda \quad (1c)$$

The boundary conditions (1a) and (1b) assume respectively the value of temperature at point $\mathbf{p}(x_p, y_p)$ on boundary (L_q) (Dirichlet boundary condition), and the value of heat flux at any point $\mathbf{p}(x_p, y_p)$ on boundary (L_T) (Neumann boundary condition). The initial condition (1c) assumes the value of temperature at point $\mathbf{v}(x_v, y_v)$ inside the domain at initial time $t=0$.

Particular form of the boundary problem for transient heat equation (1) is the formulation with the condition of periodic changes of the temperature on the boundary, which takes the following form:

$$T(x, y, t) = \tilde{T} \cos(\omega t) \quad , \quad (x, y) \in (L) \quad (1d)$$

where \tilde{T} is the amplitude of the temperature oscillations.

The sketch for two dimensional boundary problem analysis of Fourier equation (1) is shown in Fig. 1.

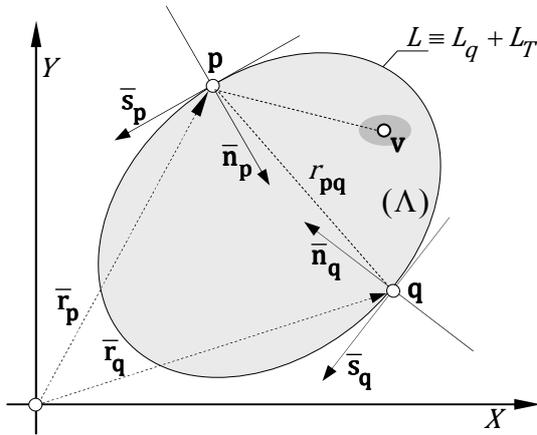


Fig. 1. Sketch for the two dimensional boundary problem analysis of Fourier equation

3. PROBLEM FORMULATION

The fundamental solution of heat conductivity equation (1), also called Green function for heat equation, and its normal derivative for two dimensional problems are given by:

$$T^*(\mathbf{p}, \mathbf{q}; t, \tau) = \frac{1}{4\pi\alpha(t-\tau)} \exp\left[-\frac{r_{\mathbf{p}\mathbf{q}}^2}{4\alpha(t-\tau)}\right] \quad (2a1)$$

$$Q^*(\mathbf{p}, \mathbf{q}; t, \tau) = \frac{|r_{\mathbf{p}\mathbf{q}} \cos(r_{\mathbf{p}\mathbf{q}}, \bar{\mathbf{n}}_{\mathbf{q}})|}{8\pi\alpha^2(t-\tau)^2} \exp\left[-\frac{r_{\mathbf{p}\mathbf{q}}^2}{4\alpha(t-\tau)}\right] \quad (2a2)$$

The solution of the Fourier first problem in closed domain (Λ) is described by the sum of double layer heat potential and Poisson-Weierstrass integral:

$$T(\mathbf{p}, t) + \alpha \int_{t_0}^t \int_{(L_q)} T(\mathbf{q}, \tau) Q^*(\mathbf{p}, \mathbf{q}; t, \tau) dL_{\mathbf{q}} d\tau + \iint_{(\Lambda)} T_0(\mathbf{v}) T^*(\mathbf{p}, \mathbf{v}; t, \tau_0) d\Lambda_{\mathbf{v}} = 0 \quad (3)$$

Density $T(\mathbf{q}, \tau)$ of double layer potential satisfies integral equation on boundary line (L):

$$-\frac{1}{2} T(\mathbf{p}, t) + \alpha \int_{t_0}^t \int_{(L_q)} T(\mathbf{q}, \tau) Q^*(\mathbf{p}, \mathbf{q}; t, \tau) dL_{\mathbf{q}} d\tau = g(\mathbf{p}, t) \quad (3a)$$

where:

$$g(\mathbf{p}, t) = T_L(\mathbf{p}, t) - \iint_{(\Lambda)} T_0(\mathbf{v}) T^*(\mathbf{p}, \mathbf{v}; t, \tau) d\Lambda_{\mathbf{v}}. \quad (3b)$$

The solution of the Fourier second problem in closed domain: (Λ) is described by the sum of single layer heat potential and Poisson-Weierstrass integral:

$$q(\mathbf{p}, t) + \frac{1}{c} \int_{t_0}^t \int_{(L_T)} q(\mathbf{q}, \tau) T^*(\mathbf{p}, \mathbf{q}; t, \tau) dL_{\mathbf{q}} d\tau + \iint_{(\Lambda)} T_0(\mathbf{v}) T^*(\mathbf{p}, \mathbf{v}; t, \tau_0) d\Lambda_{\mathbf{v}} = 0 \quad (4)$$

Density $q(\mathbf{q}, \tau)$ of single layer potential satisfies integral equation on boundary line (L):

$$\frac{1}{2} q(\mathbf{p}, t) - \frac{1}{c} \int_{t_0}^t \int_{(L_T)} q(\mathbf{q}, \tau) T^*(\mathbf{p}, \mathbf{q}; t, \tau) dL_{\mathbf{q}} d\tau = h(\mathbf{p}, t) \quad (4a)$$

where:

$$h(\mathbf{p}, t) = q_L(\mathbf{p}, t) - \iint_{(\Lambda)} T_0(\mathbf{v}) T^*(\mathbf{p}, \mathbf{v}; t, \tau) d\Lambda_{\mathbf{v}}. \quad (4b)$$

3.1. Boundary integral equation for heat conduction equation

The mixed internal Fourier problem for differential equation (1) with conditions (1a,1b) and (1c) in two dimensional area (Λ) has the general solution of the integral form (Brebbia et al, 1984)

$$\chi(\mathbf{p}) T(\mathbf{p}, t_k) + \alpha \int_{t_0}^{t_k} \int_{(L_q)} T(\mathbf{q}, \tau) Q^*(\mathbf{p}, \mathbf{q}; t_k, \tau) dL_{\mathbf{q}} d\tau + \frac{1}{c} \int_{t_0}^{t_k} \int_{(L_T)} q(\mathbf{q}, \tau) T^*(\mathbf{p}, \mathbf{q}; t_k, \tau) dL_{\mathbf{q}} d\tau + \iint_{(\Lambda)} T_0(\mathbf{v}) T^*(\mathbf{p}, \mathbf{v}; t_k, \tau_0) d\Lambda_{\mathbf{v}} = 0, \quad (5)$$

where \mathbf{p} and \mathbf{q} , \mathbf{v} are respectively source and field points within the domain (Λ) or on the boundary (see Fig.1) and $[t_0, t_k]$ is the analyzed time interval. Coefficient $\chi(\mathbf{p})$ is related to the local geometry of the boundary at point (\mathbf{p}). For smooth boundary point $\chi(\mathbf{p})=1/2$ and for an internal point $\chi(\mathbf{v})=1$.

Unknown functions in integral equation (5) are: temperature $T(\mathbf{q}, \tau)$ on the part (L_q) of boundary line and heat flux on the part (L_T) of boundary line, whereat $L = L_q \cup L_T$.

In the simplest method of discretization the integral equation (5) in relation to time, it is supposed that the time variations of the functions $T(\mathbf{q}, \tau)$ and $q(\mathbf{q}, \tau)$ are small as compared to functions $T^*(\mathbf{p}, \mathbf{q}; t_k, \tau)$ and $Q^*(\mathbf{p}, \mathbf{q}; t_k, \tau)$. It can be reasonably assumed that the functions $T(\mathbf{q}, \tau)$ and $q(\mathbf{q}, \tau)$ are also constant in small time period $[t_{k-1}, t_k]$ (Wrobel, 2002; Kythe, 2005).

Accordingly to the above assumption the integral equation (5) can be denoted in the form:

$$\begin{aligned} \chi(\mathbf{p})T(\mathbf{p}, t_k) + \alpha \int_{(L_q)} T(\mathbf{q}, \tau) \tilde{Q}^*(\mathbf{p}, \mathbf{q}; t_k, \tau) dL_{\mathbf{q}} + \\ \frac{1}{c} \int_{(L_T)} q(\mathbf{q}, \tau) \tilde{T}^*(\mathbf{p}, \mathbf{q}; t_k, \tau) dL_{\mathbf{q}} + \\ \iint_{(\Lambda)} T_0(\mathbf{v}) \hat{T}^*(\mathbf{p}, \mathbf{v}; t_k, t_0) d\Lambda_{\mathbf{v}} = 0, \end{aligned} \quad (6)$$

where the kernels $\tilde{T}^*(\mathbf{p}, \mathbf{q}; t_k, \tau)$ and $\tilde{Q}^*(\mathbf{p}, \mathbf{q}; t_k, \tau)$ are given by expression:

$$\begin{aligned} \tilde{T}^*(\mathbf{p}, \mathbf{q}; t_k, \tau) &= \frac{1}{c} \int_{(L_T)} T^*(\mathbf{p}, \mathbf{q}; t_k, \tau) dL_{\mathbf{q}} = \\ &= \frac{1}{4\pi\lambda} \int_{t_0}^{t_k} \frac{1}{(t_k - \tau)} \exp\left[-\frac{r_{\mathbf{p}\mathbf{q}}^2}{4\alpha(t_k - \tau)}\right] d\tau = \\ &= \frac{1}{4\pi\lambda} Ei\left(-\frac{r_{\mathbf{p}\mathbf{q}}^2}{4\alpha(t_k - t_0)}\right), \end{aligned} \quad (6a)$$

where $Ei(\cdot)$ is the exponential integral function:

$$\begin{aligned} \tilde{Q}^*(\mathbf{p}, \mathbf{q}; t_k, \tau) &= \alpha \int_{(L_T)} Q^*(\mathbf{p}, \mathbf{q}; t_k, \tau) dL_{\mathbf{q}} = \\ &= \frac{d}{8\pi\alpha} \int_{t_0}^{t_k} \frac{1}{(t_k - \tau)^2} \exp\left[-\frac{r_{\mathbf{p}\mathbf{q}}^2}{4\alpha(t_k - \tau)}\right] d\tau = \\ &= \frac{d}{2\pi r_{\mathbf{p}\mathbf{q}}^2} \exp\left(-\frac{r_{\mathbf{p}\mathbf{q}}^2}{4\alpha(t_k - t_0)}\right), \end{aligned} \quad (6b)$$

where $d = r_{x\mathbf{p}\mathbf{q}} \cdot \left| \frac{\partial y}{\partial l} \right|_{\mathbf{q}} - r_{y\mathbf{p}\mathbf{q}} \cdot \left| \frac{\partial x}{\partial l} \right|_{\mathbf{q}}$.

3.2. Boundary integral equation for heat conduction equation with periodic boundary condition

The unsteady heat conduction problem in two dimensional object with condition of periodical changes of temperature on boundary line is described by the equation (1) with periodic boundary condition (1d).

In this case, the temperature may be treated as the function:

$$T(x, y, t) = U(x, y) \exp(-i\omega t) \quad (7)$$

where only the real part of the above expression has physical meaning as consequence of boundary condition (1d) and the basic relation for complex functions: $\exp(-iz) = \cos(z) - i\sin(z)$.

Inserting space and time derivatives of the temperature, expressed by relation (7), to the equation (1), leads to the modified Helmholtz differential equation for function $U \equiv U(x, y)$ (Sorko and

Karpowich, 2007).

$$\nabla^2 U - \hat{k}^2 U = 0; \quad \text{where } \hat{k} = \sqrt{-i\omega/\alpha} \quad (8)$$

The integral solution of differential equation (8) has the form:

$$\begin{aligned} \int_{(L)} G(\mathbf{p}, \mathbf{q}) \frac{\partial}{\partial n_{\mathbf{q}}} U(\mathbf{q}) dL_{\mathbf{q}} = \\ -\chi(\mathbf{p})U(\mathbf{p}) + \int_{(L)} U(\mathbf{q}) \frac{\partial}{\partial n_{\mathbf{q}}} G(\mathbf{p}, \mathbf{q}) dL_{\mathbf{q}} \end{aligned} \quad (9)$$

where Green function $G(\mathbf{p}, \mathbf{q})$ of the Helmholtz equation is given by a modified Bessel function:

$$G(\mathbf{p}, \mathbf{q}) = \frac{1}{2\pi} K_0(\hat{k}r_{\mathbf{p}\mathbf{q}}) \quad (10)$$

Modified Bessel function of order (0) of complex argument ($\hat{k}r_{\mathbf{p}\mathbf{q}}$) can be expressed by the Kelvin functions of real argument ($kr_{\mathbf{p}\mathbf{q}}$) in which $k = \sqrt{\omega/\alpha}$, then the Green function and its derivative take the form:

$$G(\mathbf{p}, \mathbf{q}) = \frac{1}{2\pi} K_0(\hat{k}r_{\mathbf{p}\mathbf{q}}) = \frac{1}{2\pi} \left[\ker_0(kr_{\mathbf{p}\mathbf{q}}) - ikei_0(kr_{\mathbf{p}\mathbf{q}}) \right] \quad (10a)$$

$$\begin{aligned} \frac{\partial}{\partial n_{\mathbf{q}}} G(\mathbf{p}, \mathbf{q}) &= \left(\frac{\partial}{\partial r} G(\mathbf{p}, \mathbf{q}) \right)_{\mathbf{q}} \left(\frac{\partial r}{\partial n} \right)_{\mathbf{q}} = H(\mathbf{p}, \mathbf{q}) \\ &= k \left[\ker_0'(kr_{\mathbf{p}\mathbf{q}}) - ikei_0'(kr_{\mathbf{p}\mathbf{q}}) \right] \left(\frac{\partial r}{\partial n} \right)_{\mathbf{q}} \end{aligned} \quad (10b)$$

Taking the limit as the source point \mathbf{p} approaching contour (L), where function $U(\mathbf{q})$ is equal to the amplitude of the temperature oscillations and expressing the limit of the double layer potential in equation (9) in terms of its principal value (when $\mathbf{p} \equiv \mathbf{q}$), one obtains integral equation of the first kind for the normal derivative of the function $U(\mathbf{q})$

$$\int_{(L)} G(\mathbf{p}, \mathbf{q}) \frac{\partial}{\partial n_{\mathbf{q}}} U(\mathbf{q}) dL_{\mathbf{q}} = \tilde{T} \left[\int_{(L)}^{PV} H(\mathbf{p}, \mathbf{q}) dL_{\mathbf{q}} - \frac{1}{2} \right] \quad (11)$$

Separating the real part and imaginary part of kernels in integral equation (10), one receives the system of two integral equations in relation to functions $F(\mathbf{q}) = (F_{Re}(\mathbf{q}), F_{Im}(\mathbf{q}))$, which are the derivatives of function $U(\mathbf{q})$.

$$\int_{(L)} G_{Re}(\mathbf{p}, \mathbf{q}) F_{Re}(\mathbf{q}) dL_{\mathbf{q}} = \tilde{T} \left[\int_{(L)}^{PV} H_{Re}(\mathbf{p}, \mathbf{q}) dL_{\mathbf{q}} - \frac{1}{2} \right] \quad (11a)$$

$$\int_{(L)} G_{Im}(\mathbf{p}, \mathbf{q}) F_{Im}(\mathbf{q}) dL_{\mathbf{q}} = \tilde{T} \left[\int_{(L)}^{PV} H_{Im}(\mathbf{p}, \mathbf{q}) dL_{\mathbf{q}} - \frac{1}{2} \right] \quad (11b)$$

After determination of discrete values $F_{Re}(\mathbf{q})$, $F_{Im}(\mathbf{q})$ on the boundary (L), the values $U_{Re}(\mathbf{q})$, $U_{Im}(\mathbf{q})$ of the function $U(\mathbf{q})$ at points of domain (Λ) are obtained from the system of equations

$$U_{Re}(\mathbf{p}) = \int_{(L)} F_{Re}(\mathbf{q}) G_{Re}(\mathbf{p}, \mathbf{q}) dL_{\mathbf{q}} + \tilde{T} \int_{(L)} H_{Re}(\mathbf{p}, \mathbf{q}) dL_{\mathbf{q}} \quad (12a)$$

$$U_{Im}(\mathbf{p}) = \int_{(L)} F_{Im}(\mathbf{q}) G_{Im}(\mathbf{p}, \mathbf{q}) dL_{\mathbf{q}} + \tilde{T} \int_{(L)} H_{Im}(\mathbf{p}, \mathbf{q}) dL_{\mathbf{q}} \quad (12b)$$

4. NUMERICAL SOLUTION OF INTEGRAL EQUATION OF HEAT CONDUCTION WITH PERIODIC BOUNDARY CONDITION

Numerical solution of integral equations in two dimensional problems consists in discretization of the boundary line into straight or arc elements with constant or linear distributed value of investigated function and consequently, the integral equation transforms to the system of algebraic linear equations in relation to the unknown integrand.

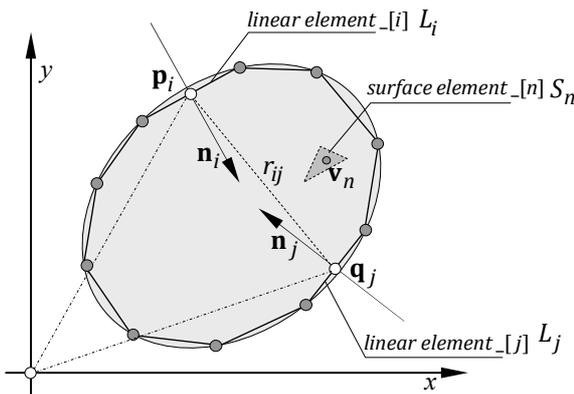


Fig. 2. Discretization of area (Λ)

Discrete solution of integral equation (6) can be obtained dividing the boundary line (L) into I ($i=1,2,3,\dots,I$), straight line elements, domain (Λ) into N ($n=1,2,3,\dots,N$) surface elements and time interval $[t_0, t_k]$ into K ($k=1,2,3,\dots,K$) subintervals (Fig. 2).

Using the nodal values, with assumption that the functions $T(\mathbf{q}_j, \tau)$ and $q(\mathbf{q}_j, \tau)$ are constant on each linear element L_j , function $T_0(\mathbf{v}_n)$ is constant on each surface element S_n and also they are constant at any subintervals $[t_{k-1}, t_k]$, the boundary integral equation is obtained in discrete form as follows:

$$\begin{aligned} \chi(\mathbf{p}_i) T(\mathbf{p}_i, t_k) + \alpha \sum_{j=1}^I \sum_{k=1}^K T(\mathbf{q}_j, t_k) \tilde{Q}^*(\mathbf{p}_i, \mathbf{q}_j; t_k, t_0) \Delta L_j + \\ \frac{1}{c} \sum_{j=1}^I \sum_{k=1}^K q(\mathbf{q}_j, t_k) \tilde{T}^*(\mathbf{p}_i, \mathbf{q}_j; t_k, t_0) \Delta L_j + \\ \sum_{n=1}^N T_0(\mathbf{v}_n) \sum_{k=1}^K \hat{T}^*(\mathbf{p}_i, \mathbf{v}_n; t_k, t_0) \Delta S_n, \end{aligned} \quad (13)$$

where:

$$\tilde{Q}^*(\mathbf{p}_i, \mathbf{q}_j; t_k, t_0) = \frac{d_{ij}}{2\pi r_{ij}^2} \exp\left(-\frac{r_{ij}^2}{4\alpha(t_k - t_0)}\right) \quad (13a)$$

$$d_{ij} = \Delta x_{ij} \cdot |\Delta y / \Delta l|_j - \Delta y_{ij} \cdot |\Delta x / \Delta l|_j \quad (13a^*)$$

$$\tilde{T}^*(\mathbf{p}_i, \mathbf{q}_j; t_k, t_0) = \frac{1}{4\pi\lambda} Ei\left(\frac{r_{ij}^2}{4\alpha(t_k - t_0)}\right) \quad (13b)$$

$$\hat{T}^*(\mathbf{p}_i, \mathbf{v}_n; t_k, t_0) = \frac{1}{4\pi\lambda} Ei\left(\frac{r_n^2}{4\alpha(t_k - t_0)}\right) \quad (13c)$$

Similarly, the integral equation (10) expressed with the system of two integral equations (10a) and (10b), describing properly the real part and the imaginary part of the function, by discretization of the boundary line moves to two systems of linear equations:

$$\sum_{j=1}^I F_{Re}(\mathbf{q}_j) \tilde{G}_{Re}(\mathbf{p}_i, \mathbf{q}_j) = \tilde{T} \left[\sum_{j=1}^I \tilde{H}_{Re}(\mathbf{p}_i, \mathbf{q}_j) - \frac{1}{2} \right] \quad (14a)$$

$$\sum_{j=1}^I F_{Im}(\mathbf{q}_j) \tilde{G}_{Im}(\mathbf{p}_i, \mathbf{q}_j) = \tilde{T} \left[\sum_{j=1}^I \tilde{H}_{Im}(\mathbf{p}_i, \mathbf{q}_j) - \frac{1}{2} \right] \quad (14b)$$

where:

$$G_{Re}(\mathbf{p}_i, \mathbf{q}_j) = \int_{(L_j)} G_{Re}(\mathbf{p}_i, \mathbf{q}_j) dL_j \quad (14a^*)$$

$$\tilde{H}_{Re}(\mathbf{p}_i, \mathbf{q}_j) = \int_{(L_j)} H_{Re}(\mathbf{p}_i, \mathbf{q}_j) dL_j$$

$$\tilde{G}_{Im}(\mathbf{p}_i, \mathbf{q}_j) = \int_{(L_j)} G_{Im}(\mathbf{p}_i, \mathbf{q}_j) dL_j \quad (14b^*)$$

$$\tilde{H}_{Im}(\mathbf{p}_i, \mathbf{q}_j) = \int_{(L_j)} H_{Im}(\mathbf{p}_i, \mathbf{q}_j) dL_j$$

5. EXAMPLES

Basing on developed BEM algorithm, a new authoring computer program was written in Fortran, which was applied to the following examples.

Example 1:

The accuracy of the formulation was tested by computing the heat field in a finite square $a=1.0$ m, when non zero initial temperatures are prescribed inside the domain and variable temperatures are assumed on the boundaries.

The thermal properties of the homogeneous medium are assumed to be: thermal conductivity $\lambda=200.0$ W/(mK), volumetric

specific heat $c=2.0 \cdot 10^6 \text{ J/(m}^3\text{K)}$, which defines a thermal diffusivity $\alpha=1.0 \cdot 10^{-4} \text{ m}^2/\text{s}$.

The temperature distribution on the boundary at $t_0=0$ is described by the relations:

$$\left. \begin{aligned} T(1, y, 0) &= 100.0 \cdot (1.0 - \sin(0.5\pi y)) \\ T(x, 1, 0) &= 100.0 \cdot (1.0 - \cos(0.5\pi x)) \\ T(0, y, 0) &= 100.00 \\ T(x, 0, 0) &= 100.00 \end{aligned} \right\} \quad (15a)$$

and the initial temperature distribution, satisfying the boundary conditions described above, is given by the relation:

$$T(x, y, 0) = 100.0 \cdot (1.0 - \cos(0.5\pi x)) \sin(0.5\pi y) \quad (15b)$$

and is presented on the sketch (Fig. 3b.)

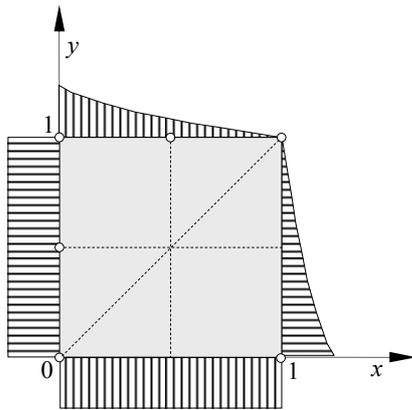


Fig. 3a. The unit square and boundary conditions

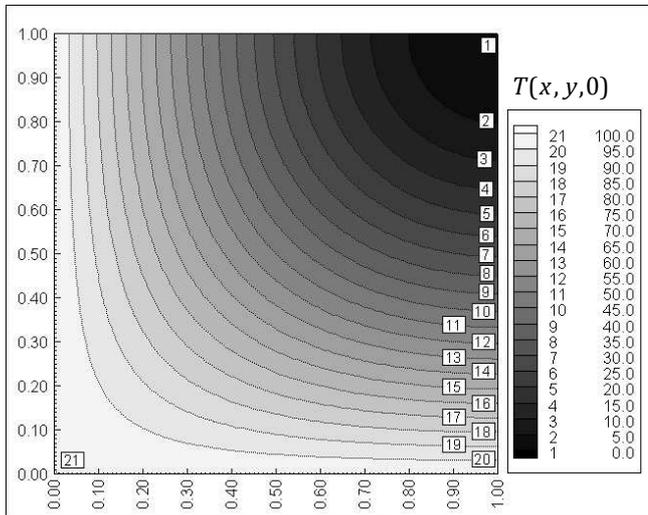


Fig. 3b. Temperature distribution in the unit square

The field of the temperature is symmetrical in relation to the diagonal of the square, so the time changes of the temperature can conveniently be presented along the line $x=y$.

Temporal evolution of the temperature $T=f(t)$ along diagonal of the square is shown on sketch (Fig. 4.) and the changes of field temperature are presented on sketch (Fig. 5.)

The presented above problem has the analytical solution as follows

$$T(x, y, t) = 100 \cdot \left[1 - \exp(-2\pi^2 \alpha t) \cos\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi y}{2}\right) \right] \quad (15c)$$

In numerical solution of considered problem with the boundary element method one assumes: 400 similar line elements on boundary, 100 square elements with collocation points in the geometrical center of every area the temporary step $\Delta t = 1.0$ at estimated time $t_{max}=3600$.

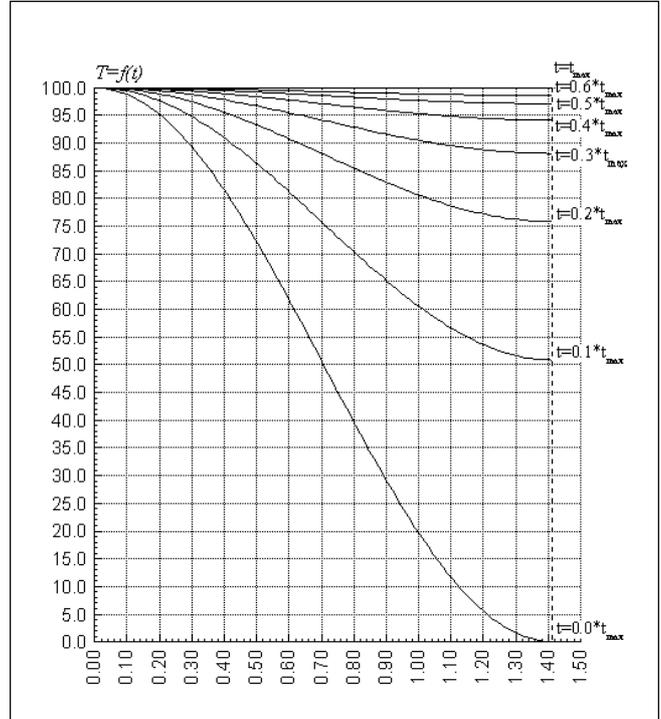


Fig. 4. Temperature distribution $T=f(t)$ along diagonal of the square

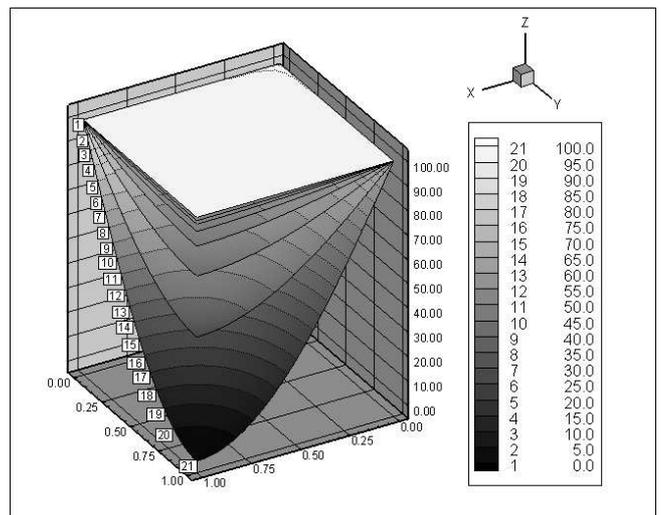


Fig. 5. Temperature distribution $T=f(t)$ in the square

The maximum error of numeric solution, estimated from relation:

$$\delta_{max} = 100(T(x, y, t)_{th} - T(x, y, t)_{num}) / T(x, y, t)_{th} \quad (16)$$

does not exceed the value 0.1%.

Example 2:

In technical problems of optimization the devices using renewable thermal energy, that for designing ground heat exchangers (horizontal and vertical) of heat pump systems, it is necessary to determine the annual ground temperature distribution for various values of ground thermal conductivity coefficient. This problem is the subject of many empirical studies, leading to formulation of complex empirical formulas describing the annual temperature propagation.

The mathematical description of ground temperature distribution problem consists in solving the transient heat conduction problem in homogeneous or heterogeneous area with constant thermal conductivity coefficient and with boundary conditions assuming the heat flux on the boundaries of value equal 0 (Fig. 6). On the ground surface the boundary periodic condition is assumed, that is the changeable annual temperature of ambient in the form:

$$T_u = T_{sa} + \Delta T_a \cos(\omega t) \tag{17}$$

The problem can be considered in heterogeneous area composed of layers of known thickness and known values of thermal conductivity coefficient.

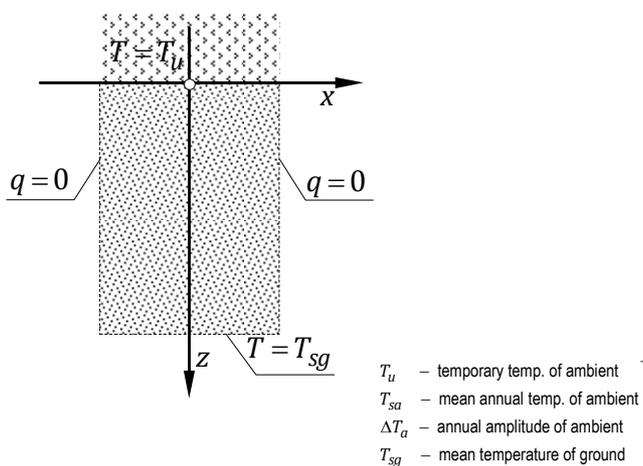


Fig. 6. Area with boundary temperature and heat flux

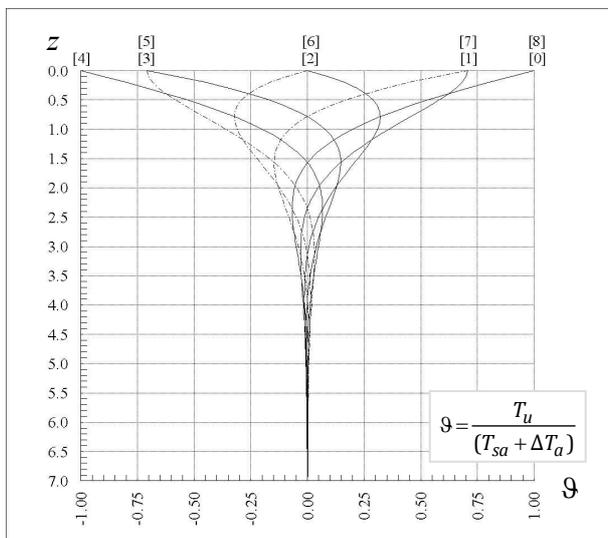


Fig. 7. Temperature profile in the ground

The Fig. 6 shows the sketch of area with boundary conditions of considered problem. The solution of the problem, in the form of unified temperature distribution (8) from the surface layer to layer of constant temperature at every moment of cycle of annual changes of temperature, is presented in the Fig. 7.

6. CONCLUSIONS

In this paper the utility of the boundary element method for solving the transient heat conduction problem with periodic boundary condition is proved. The general solutions of Fourier equation with initial and boundary value problems are introduced, on the assumption that temperature changes periodically on the boundary. The new mathematical algorithm is developed, which is further verified by solving transient heat conduction problem in two dimensional area. The comparison between analytical and numerical solution of test problem proves the great accuracy of proposed BEM algorithm. Finally, the method is applied to solve the ground temperature distribution problem with the boundary condition of the oscillating temperature of ambient. All numerical computations were made with the use of a new computer program, written by authors, in Fortran.

Although, the boundary element method is not so widely applied, as an efficient numerical method and computational tool, constitutes the great alternative to popular mesh methods (FEM, FDM), and can be successfully employed for analysis of many engineering problems.

REFERENCES

1. **Bialecki R.A., Jurgaś P., Günther K.** (2002), Dual reciprocity BEM without matrix inversion for transient heat conduction, *Engineering Analysis with Boundary Elements*, 26, 227–236.
2. **Brebbia C.A.** (ed) (1984) Topics in Boundary Element Research. Vol1. Basic Principles and Applications Springer-Verlag
3. **Brebbia C.A., Telles J.C.F., Wrobel L.C.** (1984), Boundary Element Techniques. Theory and Applications in Engineering, Springer-Verlag
4. **Cheng R.J., Liew K.M.** (2012), A meshless analysis of three-dimensional transient heat conduction problems, *Engineering Analysis with Boundary Elements* 36, 203–210.
5. **Erhart K., Divo E., Kassab A.J.** (2006), A parallel domain decomposition boundary element method approach for the solution of large-scale transient heat conduction problems, *Engineering Analysis with Boundary Elements*, 30, 553–563.
6. **Godinho L., Tadeu A., Simoes N.** (2004), Study of transient heat conduction in 2.5D domains using the boundary element method, *Engineering Analysis with Boundary Elements* 28, 593–606.
7. **Johansson B.T., Lesnic D.** (2008), A method of fundamental solutions for transient heat conduction, *Engineering Analysis with Boundary Elements*, 32, 697–703.
8. **Johansson T., Lesnic D.** (2009), A method of fundamental solutions for transient heat conduction in layered materials, *Engineering Analysis with Boundary Elements*, 33, 1362–1367.
9. **Katsikadelis, J.T.** (2002), *Boundary Elements. Theory and Applications*, Elsevier Science Ltd.
10. **Kythe P.K.** (2005), *Introduction to Boundary Element Methods*, CRC Press.
11. **Li Q.-H., Chen S.-S., Kou G.-X.** (2011), Transient heat conduction analysis using the MLPG method and modified precise time step integration method, *Journal of Computational Physics*, 230, 2736–2750.
12. **Lu X., Tervola P., Viljanen M.** (2006), Transient analytical solution to heat conduction in composite circular cylinder, *International Journal of Heat and Mass Transfer*, 49, 341–348.

13. **Lu X., Viljanen M.** (2006), An analytical method to solve heat conduction in layered spheres with time-dependent boundary conditions. *Physics Letters A* 351, 274–282.
14. **Majchrzak E.** (2001), *Boundary element method in heat transfer*, Częstochowa University of Technology (in Polish).
15. **Mansur W.J., Vasconcellos C.A.B., Zambrozuski N.J.M., Rotunno Filho O.C.** (2009), Numerical solution for the linear transient heat conduction equation using an Explicit Green's Approach. *International Journal of Heat and Mass Transfer*, 52, 694–701.
16. **Mohammadia M., Hematiyan M.R., Marin L.** (2010), Boundary element analysis of nonlinear transient heat conduction problems involving non-homogenous and nonlinear heat sources using time-dependent fundamental solutions, *Engineering Analysis with Boundary Elements*, 34, 655–665.
17. **Monte F., Beck J.V., Amos D.E.** (2012), Solving two-dimensional Cartesian unsteady heat conduction problems for small values of the time, *International Journal of Thermal Sciences* 60, 106–113.
18. **Ochiai Y., Kitayama Y.** (2009) Three-dimensional unsteady heat conduction analysis by triple-reciprocity boundary element method, *Engineering Analysis with Boundary Elements*, 33, 789–795.
19. **Ochiai Y., Sladek V., Sladek J.** (2006) Transient heat conduction analysis by triple-reciprocity boundary element method, *Engineering Analysis with Boundary Elements*, 30, 194–204.
20. **Pozrikidis C.A.** (2000), *Practical Guide to Boundary Element Methods with the software*, Library BEMLIB Chapman&Hall/CRC.
21. **Rantala J.** (2005), A new method to estimate the periodic temperature distribution underneath a slab-on-ground structure, *Building and Environment*, 40, 832–840.
22. **Simoes N., Tadeu A., Antonio J., Mansur W.** (2012), Transient heat conduction under nonzero initial conditions: A solution using the boundary element method in the frequency domain, *Engineering Analysis with Boundary Elements*, 36, 562–567.
23. **Singh S., Jain P.K., Rizwan-uddin** (2008), Analytical solution to transient heat conduction in polar coordinates with multiple layers in radial direction, *International Journal of Thermal Sciences*, 47, 261–273.
24. **Soleimani S., Jalaal M., Bararnia H., Ghasemi E., Ganji D.D., Mohammadi F.** (2010), Local RBF-DQ method for two-dimensional transient heat conduction problems, *International Communications in Heat and Mass Transfer*, 37, 1411–1418.
25. **Sorko S.A., Karpovich S.** (2007), Solving the unsteady heat transfer problem with periodic boundary condition by the boundary integral equations method, *Teoretičeskaä i Prikladnaä Mehanika*, Vol. 22.
26. **Sutradhar A., Paulino G.H.** (2004), The simple boundary element method for transient heat conduction in functionally graded materials, *Computer Methods in Applied Mechanics and Engineering*, 193, 4511–4539.
27. **Tanaka M., Matsumoto T., Takakuwa S.** (2006), Dual reciprocity BEM for time-stepping approach to the transient heat conduction problem in nonlinear materials, *Computer Methods in Applied Mechanics and Engineering*, 195, 4953–4961.
28. **Wrobel L.C.** (2002), *The Boundary Element Method Vol I. Applications in Thermo-Fluids and Acoustic*, Willey.
29. **Yang K., Gao X.-W.** (2010), Radial integration BEM for transient heat conduction problems, *Engineering Analysis with Boundary Elements*, 34, 557–563.
30. **Yumrutas R., Unsal M., Kanoglu M.** (2005), Periodic solution of transient heat flow through multilayer walls and flat roofs by complex finite Fourier transform technique, *Building and Environment*, 40, 1117–1125.
31. **Zhang X.H., Ouyang J., Zhang L.** (2009), Matrix free meshless method for transient heat conduction problems, *International Journal of Heat and Mass Transfer*, 52, 2161–2165.

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