FUZZY PREDICTIVE CONTROL OF FRACTIONAL-ORDER NONLINEAR DISCRETE-TIME SYSTEMS

Stefan DOMEK

Abstract: At the end of the 19th century Liouville and Riemann introduced the notion of a fractional-order derivative, and in the latter half of the 20th century the concept of the so-called Grünwald-Letnikov fractional-order derivative has been put forward. In the paper a predictive controller for MIMO fractional-order discrete-time systems is proposed, and then the concept is extended to nonlinear processes that can be modelled by Takagi-Sugeno fuzzy models. At first nonlinear and linear fractional-order discrete-time dynamical models are described. Then a generalized nonlinear fractional-order TS fuzzy model is defined, for which equations of a predictive controller are derived.

1. INTRODUCTION

The effectiveness of nonlinear process control systems depends to a large extent on the quality of the model used for controller synthesis or tuning. Unfortunately, the choice of an adequate model for a nonlinear process and its parameterization involves difficulties in industrial practice. Therefore, nonlinear process models of relatively simple structure but furnishing a means to synthesize the controller that would ensure satisfactory control performance are still looked for (Domek, 2006).

One of the more effective methods employed to describe real properties exhibited by many industrial processes, inclusive of those with distributed parameters, seems to be the description based on fractional-order derivatives. Many examples illustrating possible applications of such a description may be found in the literature (Domek and Jaroszewski, 2010; Kaczorek, 2009; Lorenz and Kastuński, 2010; Podlubny et al., 1997; Riewe, 1997; Sierociuk, 2007; Stojberg and Kari, 2002; Suarez et al., 2003; Vinagre and Feliu, 2002; Xue and Chen, 2002; Zamani et al., 2007).

In the paper a way of modelling complex nonlinear MIMO processes in state space by means of fuzzy Takagi-Sugeno models of fractional order (Domek, 2006; Takagi and Sugeno, 1985; Tatjewski, 2007) is presented and a generalized predictive algorithm that employs such models is introduced.

2. DYNAMICAL MODELS OF FRACTIONAL ORDER

Let us consider the traditional discrete-time nonlinear process model of integer order in state space, well-known in the form
\[
x(t + 1) = f(x(t), u(t)) \\
y(t) = g(x(t))
\]
where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( y(t) \in \mathbb{R}^p \) denote the state, input and output vectors respectively at time instant \( t \in \{0,1,2,\ldots\} \) of dimensions \( n \times 1 \), \( m \times 1 \) and \( p \times 1 \) respectively.

Equation (1) can be rewritten with the use of the so-called first-order backward difference for the state \( x(t) \):
\[
\Delta^1 x(t) = x(t) - x(t - 1)
\]
as
\[
\Delta^1 x(t + 1) = f_d(x(t), u(t))
\]
where
\[
f_d(x(t), u(t)) = f(x(t), u(t)) - x(t)
\]

Now, let us introduce the definition of the real fractional-order \( \alpha \) backward difference for the state vector \( x(t) \), based on the Grünwald-Letnikov definition (Sierociuk, 2007):
\[
\Delta^\alpha x(t) = \sum_{i=0}^{n-1} (-1)^i (\frac{\alpha}{\alpha-1}) x(t - \frac{i}{\alpha})
\]

The definition (6) may be written in a generalized form by adopting different orders of backward differences for individual state variables of the state vector \( x(t) \in \mathbb{R}^n \):
\[
\Delta^\alpha x(t + 1) = \begin{bmatrix} \Delta^{\alpha_1} x_1(t + 1) & \cdots & \Delta^{\alpha_n} x_n(t + 1) \end{bmatrix}^T
\]

Then, similarly to eq. (4), the fractional-order generalized model of a nonlinear process may be defined in state space as:
\[
\Delta^\alpha x(t + 1) = f_d(x(t), u(t))
\]
which, in view of eqs. (6), (7) and (8), may be rewritten in the following form:
\[
x(t + 1) = f_d(x(t), u(t)) - \sum_{i=0}^{n-1} (-1)^i Y_i x(t + 1 - i)
\]

It should be noted that the model (10), (11) in particular may describe properties of a fractional-order linear process. By analogy to the integer-order model (1) – (5), for which the well-known linear version has the form:
3. FRACTIONAL-ORDER NONLINEAR FUZZY MODELS

The usefulness of fractional-order models (9) in industrial practice, where processes to be controlled are most often significantly nonlinear, is small. Making use of the nonlinear model (10) is not possible in general, since universal methods for nonlinear controller synthesis based on such models are lacking.

On the other hand, the linear model (16) may be employed to synthesize a nonlinear process controller only if the assumption is made that the control system is operated in a small vicinity of the equilibrium point, for which the model has been defined.

The approach to modelling integer-order nonlinear processes that has been employed for many years is the use of the so-called networks (batteries) of Takagi-Sugeno local models (also called Takagi-Sugeno-Kanga models), originally proposed in Takagi and Sugeno (1985). In these models an antecedent in the form of a fuzzy logical product for each i-th fuzzy rule out of p rules is adopted (Domek, 2006; Tatjewski, 2007):

\[ x(t + 1) = A_d x(t) + B_d u(t) \]
\[ y(t) = C_d x(t) \]

where

\[ A_d = A - I_n \]

\[ B_d = B \]

\[ C_d = C \]

\[ A \in \mathbb{R}^{n \times n} \text{ - identity matrix} \]

\[ B \in \mathbb{R}^{n \times m} \]

\[ C \in \mathbb{R}^{p \times n} \]

\[ x(t) \in \mathbb{R}^n \]

\[ u(t) \in \mathbb{R}^m \]

\[ y(t) \in \mathbb{R}^p \]

\[ n \] is the order of the model

\[ m \] is the number of inputs

\[ p \] is the number of outputs

\[ A, B, \text{ and } C \] are real matrices

\[ x, u, \text{ and } y \] are real vectors

\[ I_n \] is the identity matrix of order n

\[ \alpha_{i,j} \] is the membership function of the j-th rule

\[ \mu_{S_{i,k}}(x_k(t)) \] is the membership function of the i-th antecedent

\[ \sum_{j=1}^{n} \mu_{S_{i,j}}(x_k(t)) \geq 0 \]

\[ \sum_{j=1}^{n} \alpha_{i,j} \mu_{S_{i,j}}(x_k(t)) \geq 0 \]

The consequents of rules in the Takagi-Sugeno models are given by algebraic expressions. For the considered in the paper case of a battery of fractional-order models the following consequents are suggested:

\[ x(t + 1) = A_d x(t) + B_d u(t) - \sum_{i=1}^{n} (-1)^i Y_{i,j} x(t + i - 1) \]
\[ y(t) = C_d x(t) \]

where

\[ Y_{i,j} = \text{diag } \left( \alpha_{i,j} \right) \]

and subscript j denotes a set of parameters of the j-th local model described by a complemented state matrix \( A_{d,j} \), input matrix \( B_j \) and output matrix \( C_j \) and of fractional orders \( \{ \alpha_{1,j}, \alpha_{2,j}, ..., \alpha_{n,j} \} \).

It may be shown (Domek, 2006; Tatjewski, 2007) that the following resultant state equation for the entire network of local models is obtained after performing inference and defuzzification by means of the center of gravity technique (Babuška and Verbrugen, 1996; Domek, 2006):

\[ x(t + 1) = \sum_{j=1}^{p} w_j(t) x(t + 1) \]
\[ y(t) = \sum_{j=1}^{p} w_j(t) y_i(t) \]

where weight coefficients determining the so-called degree of activation of individual rules are defined by means of the fuzzy product operator (Babuška and Verbrugen, 1996):

\[ w_j(t) = \prod_{k=1}^{n} \mu_{S_{j,k}}(x_k(t)) \]

With eqs. (22) – (24) in view, the fractional-order nonlinear process (9) may be described in state space by a fuzzy-tuned fractional-order quasi-linear model of the form (17) creating a fuzzy network of Takagi-Sugeno (TS) local models:

\[ x(t + 1) = A_d x(t) + B_d u(t) - \sum_{i=1}^{n} (-1)^i Y_{i,j} x(t + i - 1) \]
\[ y(t) = C_d x(t) \]

where

\[ A_d = A_d(x(t)) \]

\[ B_d = B_d(x(t)) \]

\[ C_d = C_d(x(t)) \]

\[ x(t) \in \mathbb{R}^n \]

\[ u(t) \in \mathbb{R}^m \]

\[ y(t) \in \mathbb{R}^p \]

\[ A_d, B_d, \text{ and } C_d \] are fuzzy sets

\[ x(t) \] is the state vector

\[ u(t) \] is the input vector

\[ y(t) \] is the output vector

\[ A, B, \text{ and } C \] are real matrices

\[ x, u, \text{ and } y \] are real vectors

\[ n \] is the order of the model

\[ m \] is the number of inputs

\[ p \] is the number of outputs

\[ n \] is the number of fuzzy sets

To identify the parameters and the order of fractional-order local dynamic models (20), (21), use can be made, for example, of Sierociuk (2007), Sierociuk and Dzieliński (2006). In Wnuk (2004) there are many remarks to be found among others, in Domek and Jaroszewski (2010), Muddu...
where
\[ Y^r(t) = \begin{bmatrix} y^r(t + N_1 t) \\ y^r(t + N_1 + 1 t) \\ \vdots \\ y^r(t + N_2 t) \end{bmatrix} \]
denotes the reference trajectory vector, which starts always from the current value of the plant output and has the form of a smoothed reference signal,
\[ Y(t) = \begin{bmatrix} y(t + N_1 t) \\ y(t + N_1 + 1 t) \\ \vdots \\ y(t + N_2 t) \end{bmatrix} \]
is the output prediction vector, whereas the vector \( \Delta \hat{U}(t) \) can take the following forms depending on the used algorithm version:
\[ \Delta \hat{U}(t) = \begin{bmatrix} \Delta_0 u(t|t) \\ \Delta_0 u(t + 1 t) \\ \vdots \\ \Delta_0 u(t + N_u - 1 t) \end{bmatrix} \]
with increments of the manipulated variable related to the component determined for the \( t \)-1 instant
\[ \Delta_0 u(t + j | t) = u(t + j | t) - u(t - 1) \],
\[ 0 \leq j \leq N_u - 1 \] (35)
with an additional assumption respectively
\[ \Delta_0 u(t + j | t) = 0, \text{ for } N_u \leq j \leq N_2 - 1 \] (36)

4.1. Synthesis of a fractional-order linear predictive controller

To determine the optimal manipulated variable we have to find the dependence of the process output prediction vector (33) on the vector of future manipulated variables (34). For the process defined by the model (17) the solution is given by Sierociuk (2007):
\[ x(t) = \Phi^Y(t)x(0) + \sum_{i=0}^{t-1} \Phi^Y(i)Bu(t - i - 1), \]
t = 1,2,... (37)
where the matrix \( \Phi^Y(t) \) is defined by the recurrence relation
\[ \Phi^Y(t + 1) = (A_0 + Y_1) \Phi^Y(t) - \sum_{i=2}^{t+1} (-1)^i Y_i \Phi^Y(t - i + 1), \]
i = 2,3,... (38)
with
\[ \Phi^Y(1) = (A_0 + Y_1), \quad \Phi^Y(0) = I_n \] (39)
Hence, in view of eqs. (13) and (37), we get
\[ y(t) = C[\Phi^Y(t)x(0) + \sum_{i=0}^{t-1} \Phi^Y(i)Bu(t - i - 1)] + D \Delta u(t) \] (40)
and consequently, assuming for simplicity \( D = 0 \), the prediction of the output for the \( t+j \) instant may be found at the \( t \) instant in the following form:
\[ y(t + j | t) = C[\Phi^Y(j)x(t) + \sum_{i=0}^{j-1} \Phi^Y(i)Bu(t + j - i - 1)] \] (41)
or equivalently
\[ y(t + j | t) = C[\Phi^Y(j)x(t) + \sum_{i=0}^{j-1} \Phi^Y(j - i - 1)Bu(t + i)] \] (42)
Hence, the prediction of the natural process response becomes
\[ y^0(t + j | t) = C[\Phi^Y(j)x(t) + \sum_{i=0}^{j-1} \Phi^Y(i)Bu(t - i)] \] (43)
and that of the forced response becomes
\[ y^r(t + j | t) = C[\sum_{i=0}^{j-1} \Phi^Y(j - i - 1)Bu(t + i)] \] (44)
Writing the future values of the natural response (43) within the prediction horizon in the vector form
\[ y^0(t) = \begin{bmatrix} y^0(t + N_1 t) \\ y^0(t + N_1 + 1 t) \\ \vdots \\ y^0(t + N_2 t) \end{bmatrix} \]
\[ y^0(t) = C \begin{bmatrix} \Phi^Y(N_1) \\ \vdots \\ \Phi^Y(N_2) \end{bmatrix} x(t) + \sum_{i=0}^{N_1-1} \Phi^Y(i)Bu(t - i) \] (46)
the output prediction vector (33), in view of eqs. (43) and (44), assumes the following form:
\[ Y(t) = E \Delta \hat{U}(t) + Y^0(t) \] (47)
where the so-called process dynamics matrix for the vector (34) is given by:
\[ E = C \cdot \Phi^Y \cdot B \] (48)
with the matrix \( E \in \mathbb{R}^{n(N_2-N_1+1) \times nN_u} \):
\[ E = \begin{bmatrix} \sum_{i=0}^{N_1-1} \Phi^Y(i) & \cdots & \cdots & \cdots & 0_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & A_0 + Y_1 & \cdots & I_n \\ \sum_{i=0}^{N_1-1} \Phi^Y(i) & \cdots & \cdots & \cdots & A_0 + Y_1 \end{bmatrix} \] (49)
and the block matrices \( B \in \mathbb{R}^{nN_u \times m} \) and \( C \in \mathbb{R}^{p(N_2-N_1+1) \times (N_2-N_1+1)} \):
\[ B = \begin{bmatrix} B & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B \end{bmatrix}, \quad C = \begin{bmatrix} C & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C \end{bmatrix} \] (50)
In view of eqs. (31) and (47), the optimal control becomes [3]:
\[ \Delta U_{opt}(t) = P[Y^r(t) - y^0(t)] \] (51)
where the controller gain matrix is:
\[ P = (E^T E + \lambda I_{mN_u})^{-1} E^T \] (52)
In view of the fact that, according to the principle of the moving horizon, only the first component of the computed control vector is utilized at the given instant $t$, we get finally from eqs. (35), (51) and (52):

$$u(t|t) = u(t-1) + p [Y^r(t)|_\omega - Y^0(t)|_\omega]$$

where $p$ is the first row of the gain matrix $P$.

In the proposed predictive controller, as with the classic predictive control, it is possible to determine the optimal control in the presence of constraints imposed on the control signal, its increments and/or process output. In such a situation, in view of eqs. (46) – (50), the quadratic programming (QP) problem should be solved numerically at each step $t$ (Domek, 2006; Maciejowski, 2002).

### 4.2. A fuzzy, fractional-order, state model predictive controller

The above presented synthesis of the fractional-order linear predictive controller can be extended in a relatively simple way to nonlinear processes by employing the proposed fuzzy TS model. In such an event a fractional-order linear predictive controller is to be determined at each step for the current quasi-linear process (25), (26). Another natural approach is utilizing the above presented method to synthesize a controller for linear processes, design of local controllers is obtained in such a way.

Therefore, a fuzzy network (battery) of local controllers is obtained in such a way.

### 5. CONCLUDING REMARKS

In the paper an approach to synthesis of a fractional-order linear predictive controller for fractional-order nonlinear MIMO processes is presented. The approach is based on utilizing the proposed fuzzy TS model of the fractional-order nonlinear process. The individual state variables in the generalized model are assumed to be of different orders. A more simple case is represented by the model of identical orders, the particular case of which is the integer-order model including the linear model. In such a case the proposed predictive controller reduces to the known SMPC controller (Maciejowski, 2002).

### REFERENCES